CELLULAR CONVECTION IN A VISCOELASTIC FLUID

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The pattern of supercritical motions in convection processes has aroused the interest of many researchers [1-4]. The fundamental theoretical results on the preferred mode near the critical point may be generalized as follows.

1. Motion driven by a surface-tension gradient generates hexagonal cells [1].

2. In buoyancy-driven convection the stable mode comprises rollers if the physical properties of the fluid are independent of the temperature [2].

3. In buoyancy-driven convection for fluids having temperature-dependent physical properties the cells are hexagonal in the interval of Rayleigh numbers close to the critical [3, 4]. These hexagons transform into rollers with a further increase in the supercriticality.

In regard to the influence of elasticity of the fluid on the mode of convective motion, this problem has been totally ignored. Below, for a more complete treatment of the physical properties of the fluid we consider a temperature-dependent viscosity. For the rheological relation we adopt the Maxwell model

$$P_{ij}+\tau_0\delta P_{ij}/\delta t=\mu u_{ij},$$

in which u_{ij} is the strain-rate tensor; μ, viscosity; τ_0, relaxation time; and

$$\frac{\delta P_{ij}}{\delta t}=\frac{\partial P_{ij}}{\partial t}+u_k\frac{\partial P_{ij}}{\partial x_k}-P_{ik}\frac{\partial u_j}{\partial x_k}-P_{jk}\frac{\partial u_i}{\partial x_k}.$$

We consider an infinite horizontal layer of the fluid, heated from below. We assume that the temperature dependence of the density and viscosity is linear.

The dimensionless convection equation has the matrix form

$$\left(I-c\tau\frac{\partial}{\partial t}\right)\frac{\partial v_i}{\partial t}+\left(I+c\tau\frac{\partial}{\partial t}\right)v_j\frac{\partial v_i}{\partial x_j}=-\left(1+c\tau\frac{\partial}{\partial t}\right)\frac{\partial P}{\partial x_i}+D_{ij}v_j+$$
$$+\tau\,\mathrm{Pr}\frac{\partial v_i}{\partial t}\delta_{i0}\delta_{i3}-\Gamma\,\mathrm{Ra}\frac{\partial}{\partial x_j}(x_3v_{ij})+\Gamma\frac{\partial}{\partial x_j}(v_0v_{ij})-c\tau\,\mathrm{Pr}\frac{\partial}{\partial x_j}Y+c\Gamma\,\mathrm{Ra}\,\tau\frac{\partial}{\partial x_j}(x_3Y)-c\Gamma\tau\frac{\partial}{\partial x_j}(v_0Y), \quad (1)$$

where the following matrix notation is used:

$$v_j=\begin{vmatrix}\theta\\ u_j\end{vmatrix};\quad \frac{\partial}{\partial x_j}=\begin{vmatrix}0\\ \frac{\partial}{\partial x_j}\end{vmatrix};\quad D_{ij}=\begin{vmatrix}\nabla^2 & \mathrm{Ra}\,\delta_{3j}\\ \mathrm{Pr}\,\delta_{ij} & \mathrm{Pr}\,\nabla^2\delta_{ij}\end{vmatrix};$$
$$v_{ij}=\begin{vmatrix}0 & 0\\ 0 & u_{ij}\end{vmatrix};\quad c=I-\delta_{i0};$$

I is the unit matrix;

$$Y=u_k\frac{\partial u_{ij}}{\partial x_k}-u_{ik}\frac{\partial u_j}{\partial x_k}+u_{jk}\frac{\partial u_i}{\partial x_k};$$

$\Gamma = h\beta\gamma\mathrm{Pr}/\mathrm{Ra}$ is the dimensionless coefficient of the temperature dependence of the viscosity; τ, dimensionless relaxation time; $\mathrm{Ra} = \alpha gAh^4\pi/\mu\varkappa$, Rayleigh number; $\mathrm{Pr} = \mu/\varkappa\rho$, Prandtl number; and θ, temperature perturbation.

To solve the nonlinear equation by perturbation methods we expand it with respect to a small parameter according to [2]. The expansion of the Rayleigh number

$$\mathrm{Ra}=\mathrm{Ra}^{(0)}+\varepsilon\mathrm{Ra}^{(1)}+\varepsilon^2\mathrm{Ra}^{(2)}+\ldots \quad (2)$$

specifies the small parameter, and $\mathrm{Ra}^{(0)}$ is the critical Rayleigh number.

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Substituting the expansion (2) into (1), we obtain equations for the first three approximations, which we omit in view of the cumbersomeness of the expressions. According to [2], from the conditions for solvability of successive-approximation equations, assuming a comparatively small relative variation of the viscosity [5], we obtain an equation for $\mathrm{Ra}^{(1)}$:

$$\Pr \mathrm{Ra}^{(1)}\theta^{(1)}u_3^{(1)} = \left\langle v_i^{(1)'}, \frac{\partial v_i^{(1)}}{\partial t}\right\rangle - \Gamma \mathrm{Ra}^{(0)}\overline{u_i^{(1)'}\frac{\partial\left(\theta^{(1)}u_{ij}^{(1)}\right)}{\partial x_i}} - \tau \Pr \mathrm{Ra}^{(0)}\overline{u_3^{(1)}\frac{\partial\theta^{(1)}}{\partial t}} - 2\Pr\tau c\left\langle v_i^{(1)'}, \frac{\partial v_j^{(1)}}{\partial x_k}\frac{\partial^2 v_i^{(1)}}{\partial x_j\partial x_k}\right\rangle -$$

$$- \Gamma \mathrm{Ra}^{(0)2}\tau\left[\overline{u_i^{(1)'}u_k^{(1)}\frac{\partial}{\partial x_k}u_{i3}^{(1)}} - \overline{u_i^{(1)'}u_{ik}^{(1)}\frac{\partial u_3^{(1)}}{\partial x_k}} - \overline{2u_i^{(1)'}x_3\frac{\partial u_j^{(1)}}{\partial x_k}\frac{\partial^2 u_i^{(1)}}{\partial x_j\partial x_k}} - \overline{u_i^{(1)'}x_3\frac{\partial u_i^{(1)}}{\partial x_k}\nabla^2 u_k^{(1)}}\right],$$

where $\theta^{(1)}$, $u_j^{(1)}$, v_j are the corresponding terms of the expansion, the angle brackets designate the scalar product [2]

$$\langle v_k', v_k''\rangle = \Pr\overline{\theta'\theta''} + \mathrm{Ra}^{(0)}\overline{u_k'u_k''},$$

the overbar signifies averaging over the entire layer, and

$$\frac{\partial v_k'}{\partial x_k} = \frac{\partial v_k''}{\partial x_k} = 0.$$

Assuming that τ and Γ are small in the third approximation, in accordance with [5], for $\mathrm{Ra}^{(2)}$ we obtain

$$\Pr \mathrm{Ra}^{(2)}\theta^{(1)'}u_3^{(1)} = \left\langle v_k^{(1)'}, \frac{\partial v_i^{(2)}}{\partial t}\right\rangle - \Pr \mathrm{Ra}^{(1)}\overline{\theta^{(1)'}u_3^{(2)}} + \left\langle v_i^{(1)'}, v_k^{(1)}\frac{\partial v_i^{(2)}}{\partial x_k}\right\rangle.$$

According to [5], we write

$$\mathrm{Ra} - \mathrm{Ra}^{(0)} = \Delta\mathrm{Ra} = \mathrm{Ra}^{(1)} + \mathrm{Ra}^{(2)}. \tag{3}$$

As in [3-5], we investigate motion consisting of two Fourier components. The vertical velocity component has the form

$$u_3^{(1)} = A_{11}f(x_3)\cos kx_1\cos lx_2 + A_{02}f(x_3)\cos 2lx_2, \quad k^2 + l^2 = 4l^2 = a^2. \tag{4}$$

For $u_3^{(2)}$ we have

$$u_3^{(2)} = \sum_{ij} k_{ij}F_{11}(x_3)\cos ikx_1\cos jlx_2. \tag{5}$$

Substituting (4) and (5) into (3), we obtain the amplitude equations

$$(K - K_1)\dot{A}_{11} = EA_{11} - AA_{11}A_{02} + BA_{11}A_{02} - R_0A_{11}^3 - PA_{11}A_{02}^2; \tag{6}$$

$$(K - K_1)\dot{A}_{02} = EA_{02} - \frac{1}{4}AA_{11}^2 + \frac{1}{4}BA_{11}^2 - R_1A_{02}^2 - \frac{1}{2}PA_{11}^2A_{02}. \tag{7}$$

The coefficients K, E, A, R_0, R_1, and P coincide with the corresponding coefficients of the amplitude equations for the Newtonian case in [5]. Equations (6) and (7) differ in the presence of the additional coefficients K_1 and B.

We consider the two free boundaries of the layer. For this case we have

$$f(z) = \cos a_k^{(0)}\sqrt{2}\,x_3, \quad a_k^{(0)} = \pi/\sqrt{2}, \quad -\frac{1}{2} \leqslant x_3 \leqslant \frac{1}{2}, \tag{8}$$

where $a_k^{(0)}$ is the critical wave number.

Direct calculations show that

$$c\left\langle v_i^{(1)'}, \frac{\partial v_j^{(1)}}{\partial x_k}\frac{\partial^2 v_i^{(1)}}{\partial x_j\partial x_k}\right\rangle = 0, \quad B = \frac{1}{2}\Gamma\mathrm{Ra}^{(0)2}\tau\int_{-1/2}^{1/2}\left[-\frac{3}{a^2}f'^2f'' -\right.$$

$$- 7f'^2f + \frac{1}{a^2}ff'f''' + 2f^2f'' - 2a^2f^3 + x_3\left(a^2ff'Df - f'fDf'' - \frac{1}{2}f'^3 -\right.$$

$$\left.\left. - \frac{1}{a^2}f'f'' - ff'f'' - \frac{2}{a^2}f'^2f'' - 3a^2f^2f' + ff''Df - a^2f^2Df\right)\right]dx_3,$$

$$k_1 = \tau\Pr\mathrm{Ra}^{(0)}a^2\int_{-1/2}^{1/2}fD^2f\,dz, \quad D = 1 - \frac{1}{a^2}\frac{\partial^2}{\partial x_3^2}.$$

Calculations for $f(x_3)$ in the form (8) yield

$$B = -9.5\Gamma \mathrm{Ra}^{(0)}\tau,$$
$$k_1 = 22.21\tau \mathrm{Pr}\mathrm{Ra}^{(0)}.$$

According to [5], we define the parameter $\varkappa$:

$$\varkappa = \frac{\Delta\,\mathrm{Ra}}{\mathrm{Ra}^{(0)}} \Big/ \left(\frac{\Delta\nu}{\nu_0}\right)^2.$$

We denote

$$A = \Gamma \mathrm{Ra}^{(0)} A_1, \quad E = \mathrm{Pr}\Delta \mathrm{Ra} E_1, \quad B = -\Gamma \mathrm{Ra}^{(0)} B_1, \quad B_1 = 9.5\mathrm{Ra}^{(0)}\tau.$$

From the steady-state conditions for (6) and (7) we infer that the transition between the hexagonal and hexagonal + roller states (the plus sign implies the possible existence of both solutions) is observed for

$$\varkappa_1 = \frac{R_1}{4(2R_0 - R_1)^2} \frac{\mathrm{Pr}}{\mathrm{Ra}^{(0)}} \frac{(A_1 + B_1)^2}{E_1}; \tag{9}$$

from the hexagonal + roller state for

$$\varkappa_2 = \frac{4R_0 + R_1}{R_1} \varkappa_1.$$

It is evident from (9) that the elasticity of the fluid increases $\varkappa_1$ and $\varkappa_2$, where $\varkappa_2 > \varkappa_1$.

Consequently, the presence of the relaxation time increases the domain of supercriticality of the existence of hexagons. This conclusion is qualitatively supported by the experimental data of [6], in which a more ordered hexagonal convective structure is observed in elastic fluids than for the same degree of supercriticality in a Newtonian fluid, i.e., in elastic fluids hexagons are found in a domain of supercriticality relatively farther-removed from the upper limit of their instability.

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